The Four Subspaces

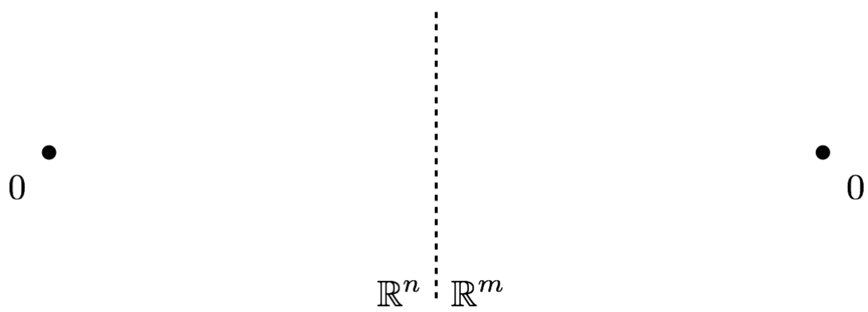
Nicholas Dwork

1 Preliminaries

Let $A \in \mathbb{R}^{m \times n}$.
Def: The Null Space of A (also called the kernel of A), denoted $N(A)$, is the set $\{x : Ax = 0\}$.
Def: The Image of A (also called the column space of A)[1], denoted $Im(A)$, is the set $\{Ax : x \in \mathbb{R}^n\}$.
Def: The dimension of a vector space is the number of elements in any basis of the vector space.
Def: The Nullity of A, denoted $\mathcal{N}(A)$, is the dimension of $N(A)$.
Def: The Rank of A, denoted $rank(A)$ is the dimension of $Im(A)$.
Def: A subset of a vector space that is itself a vector space is called a subspace.
Def: The sum of two vector spaces V, W, denoted $V + W$, is the set $\{v + w : v \in V, w \in W\}$.
Def: A vector space V is called the direct sum of two vector spaces W_1, W_2, denoted by $V = W_1 \oplus W_2$, if W_1, W_2 are subspaces of V such that $W_1 \cap W_2 = \{0\}$ and $W_1 + W_2$=V.
Def: The inner product of x, y is denoted by $\langle x, y \rangle$.
Def: Two vectors x, y are orthogonal means $\langle x, y \rangle = 0$.
Def: The dot product of two vectors $x, y \in \mathbb{R}^n$ is $x \cdot y = x^T y = x_1 y_1 + x_2 y_2 + \cdots + x_N y_N$.
Def: The orthogonal complement of a vector space V, denoted $V^\perp$, is the set $\{w : \langle v, w \rangle = 0 \; \forall \; v \in V\}$.

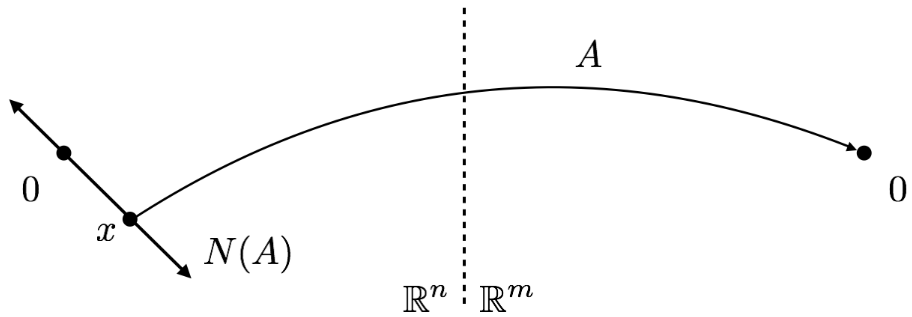
2 Euclidean Vector Spaces

We will consider a matrix $A \in \mathbb{R}^{m \times n}$. Under multiplication, $A : \mathbb{R}^n \to \mathbb{R}^m$. To aid us in our understanding of the four subspaces, we will draw both the domain and range of A side by side. The dots in the picture are the origins, denoted by 0.

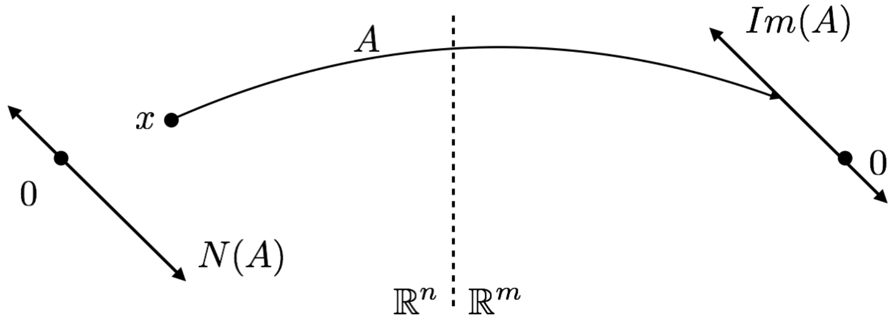


The set $N(A)$ is a subspace of $\mathbb{R}^n$. Any vector $x \in N(A)$ maps to 0 under matrix multiplication with A. Here we draw $N(A)$ as a line through the origin. Recall that any subspace of a vector space must include the origin. Any point $x \in N(A)$ is mapped to the origin of $\mathbb{R}^m$ by multiplication with A.

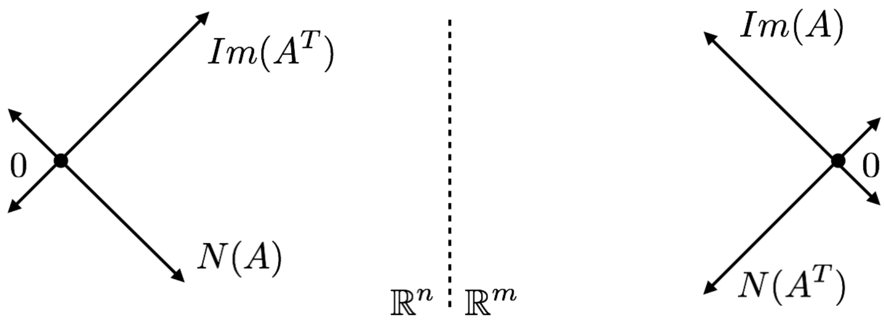
[1] It is also often called the Range of A, but I find that name confusing since it conflicts with the definition of the range of a function, and I don't recommend it.



The set $Im(A)$ is a subspace of $\mathbb{R}^m$. Any vector $x \in \mathbb{R}^n$ maps to $Im(A)$. Any vector $x \in \mathbb{R}^n - N(A)$ maps to a nonzero vector in $Im(A)$, as shown below.



Let us consider the matrix $A^T \in \mathbb{R}^{n \times m}$. Similar to our discussion of A, $N(A^T)$ is a subspace of $\mathbb{R}^m$ and $Im(A^T)$ is a subspace of $\mathbb{R}^n$.



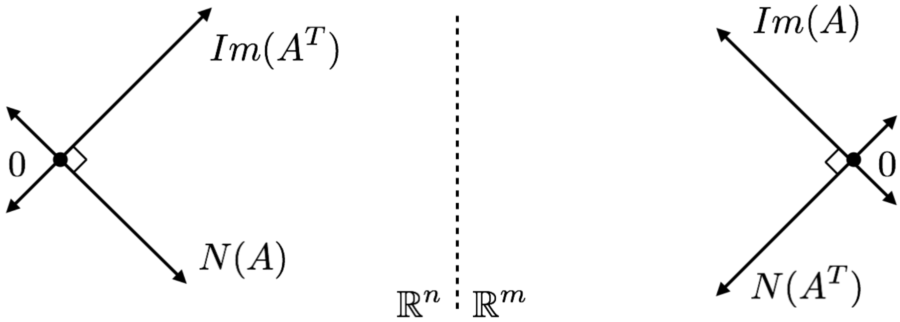
The four sets shown in the figure above, $Im(A), Im(A^T), N(A), N(A^T)$ are called the four subspaces. Denote the i^{th} column of A by a_i. Then

$$A = \begin{bmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & & | \end{bmatrix} \quad \text{and} \quad A^T = \begin{bmatrix} — & a_1^T & — \\ — & a_2^T & — \\ & \vdots & \\ — & a_n^T & — \end{bmatrix}.$$

Consider a vector $x \in N(A^T)$. This means that $A^T x = 0$. Written out more explicitly,

$$\begin{bmatrix} — & a_1^T & — \\ — & a_2^T & — \\ & \vdots & \\ — & a_n^T & — \end{bmatrix} x = \begin{bmatrix} a_1^T x \\ a_2^T x \\ \vdots \\ a_n^T x \end{bmatrix} = 0.$$

This shows that x is orthogonal to all the columns of A. Since $Im(A)$ equals the span of the columns of A, x is orthogonal to $Im(A)$. And since x was any vector in $N(A^T)$, $N(A^T) \perp Im(A)$. Similarly, $N(A) \perp Im(A^T)$.



From the above, we see that $N(A) \oplus Im(A^T) = \mathbb{R}^n$ and $N(A^T) \oplus Im(A) = \mathbb{R}^m$.

The above picture summarizes some very powerful knowledge in Linear Algebra. As an example of this power, we'll now use our new understanding to prove two very famous theorems.

The Rank-Nullity Theorem: $rank(A) + \mathcal{N}(A) = n$.
Proof: Let α be a basis for $N(A)$ with dimension a where $a \leq n$. We can find a basis β for $Im(A^T)$ of dimension $n - a$. $dim(A(\beta)) = dim(\beta)$, and $dim(A(\beta)) = dim(Im(A)) = rank(A)$. Therefore, $rank(A) = dim(\beta) = n - \mathcal{N}(A)$. ■

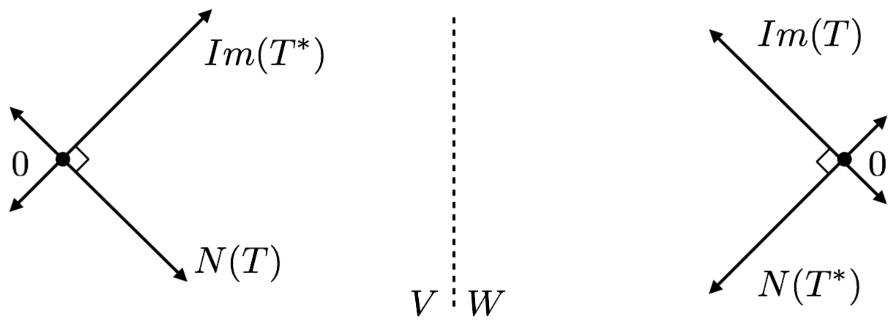
Theorem: $rank(A) = rank(A^T)$.
Proof: Since $Im(A^T) \oplus N(A) = \mathbb{R}^n$, $rank(A^T) + \mathcal{N}(A) = n$. From the last theorem, we know $rank(A) + \mathcal{N}(A) = n$. Subtracting one equation from the other yields $rank(A) - rank(A^T) = 0$. ■

The previous theorem shows us that A (when viewed as a mapping) is one-to-one when restricted to $Im(A^T)$. It's like $Im(A^T)$ is the "true" domain of A, and the vectors in $N(A)$ are just a distraction.

3 Inner Product Spaces

So far we've been working in Euclidean vector spaces. Now we'll work in the more general inner product spaces. We'll start by presenting more relevant definitions.
Def: A function f is linear means $f(x + y) = f(x) + f(y)$ and $f(kx) = kf(x)$ for any scalar k.
Def: A function T is a linear transformation means T is a linear function and it maps one vector space onto another.
Def: The adjoint of a linear transformation $T : V \to W$, denoted T^*, is the linear transformation that satisfies $\langle Tx, y\rangle = \langle x, T^*y\rangle$ for all $x \in V, y \in W$.
Def: A function T is a linear operator means T is a linear transformation that maps a vector space onto itself.
Def: The null space of $T : V \to W$, denoted $N(T)$, is the set $x \in V : T(x) = 0$.
Def: The image of $T : V \to W$, denoted $Im(T)$, is the set $\{T(x) : x \in V$.
Def: An inner product space is a vector space together with an inner product.

Let $T : V \to W$ such that T is a linear trasnformation and V, W are finite dimensional inner product spaces.
Theorem: $Im(T) \perp N(T^*)$.
Proof: Let $y \in N(T^*)$. Then $T^*(y) = 0$, which implies $\langle x, T^*y\rangle = 0$ for any $x \in V$. But $\langle x, T^*(y)\rangle = \langle T(x), y\rangle$, so $\langle T(x), y\rangle = 0$. This shows that y is perpendicular to $Im(T)$. And, since y was any vector in $N(T^*)$, the proof is complete. ■
From here, we gain an understanding like before that is summarized in the figure below.


$Im(T^*)$
0
$N(T)$
V
W
$Im(T)$
0
$N(T^*)$