Sequence Design by Signal Inversion Using EPG

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Overview

- Extended Phase Graphs (EPGs) are a computationally efficient tool for representing MR signal progression.
- The full state of the signal across a voxel is encoded into a small number of coefficients.
- There exist techniques to design a sequence using EPG, such as the "1-ahead" algorithm. But these algorithms are greedy and do not yield an optimal result.
- The purpose of this research is to use EPG to model MR signal progression as a discrete dynamical system. Further, by taking advantage of the small-tip angle approximation, we will formulate sequence design problems as constrained least-squares problems.

Extended Phase Graphs

The EPG Transform (where $n \geq 0$):

$$\begin{aligned} F_n^+ &= \int_0^1 M_{xy}(z)\, \exp\left(-i\, 2\pi\, n\, z\right)\, dz, \\ F_n^- &= \int_0^1 M^*(z)_{xy}\, \exp\left(-i\, 2\pi\, n\, z\right)\, dz, \\ Z_n &= \int_0^1 M_z(z)\, \exp\left(-i\, 2\pi\, n\, z\right)\, dz. \end{aligned}$$

The EPG coefficients are stored in a single matrix

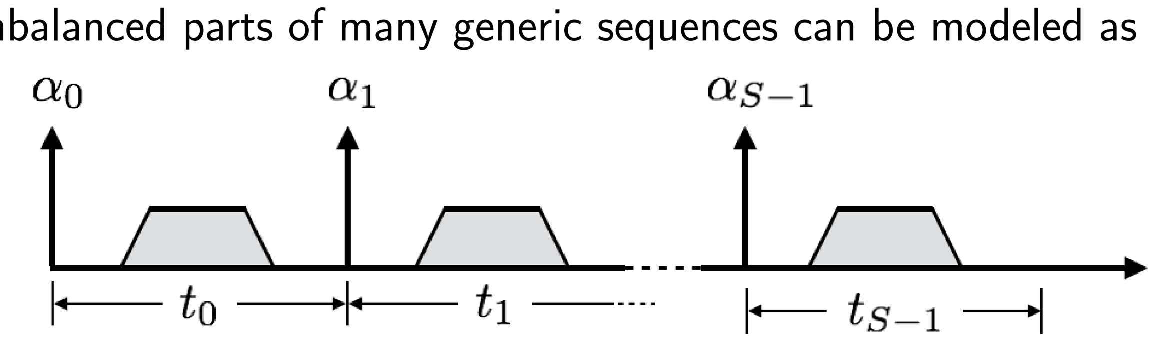
$$Q = \begin{bmatrix} F_0^+ & F_1^+ & \cdots & F_N^+ \\ F_0^- & F_1^- & \cdots & F_N^- \\ Z_0 & Z_1 & \cdots & Z_N \end{bmatrix}.$$

The synthesis equations for the EPG transform are

$$\begin{aligned} M_{xy}(z) &= F_0^+ + \sum_{n=1}^{\infty} F_n^+\, \exp\left(i\, 2\pi\, n\, z\right) + \sum_{n=1}^{\infty} F_n^-\, \exp\left(-i\, 2\pi\, n\, z\right), \\ M_z(z) &= \text{Real}\left(Z_0 + 2 \sum_{n=1}^{\infty} Z_n\, \exp\left(i\, 2\pi\, n\, z\right) \right). \end{aligned}$$

Generic Sequence

The unbalanced parts of many generic sequences can be modeled as



- α_s is the tip angle of the s^{th} RF pulse
- Shaded regions represent unbalanced G_z gradients

Discrete Dynamical System

$$Q(s+1) = E_s\, G_s\, R_{\alpha_s}\, Q(s) + d_s$$

- R_{α_s} is a 3×3 matrix for the s^{th} RF pulse
- G_s shifts the coefficients according to the s^{th} gradient
- E_s is a diagonal transformation that accounts for relaxation
- d_s accounts for recovery (Z_0 element is non-zero)

Linearizing the System

For an RF pulse applied with phase ϕ,

$$R_\alpha(\phi) = \begin{bmatrix} \cos^2(\alpha/2) & e^{i2\phi}\sin^2(\alpha/2) & ie^{i\phi}\sin(\alpha) \\ e^{-i2\phi}\sin^2(\alpha/2) & \cos^2(\alpha/2) & -ie^{-i\phi}\sin(\alpha) \\ (i/2)e^{-i\phi}\sin(\alpha) & -(i/2)e^{i\phi}\sin(\alpha) & \cos(\alpha) \end{bmatrix}.$$

When $\Delta\alpha_s$ is small,

$$R_{\alpha_s + \Delta\alpha_s} \approx R_{\alpha_s} \begin{bmatrix} 1 & 0 & \Delta\alpha_s \\ 0 & 1 & \Delta\alpha_s \\ -\Delta\alpha_s/2 & -\Delta\alpha_s/2 & 1 \end{bmatrix},$$

With this approximation, the Discrete Dynamical System becomes

$$\begin{bmatrix} F(s+1)_n^+ \\ F(s+1)_n^- \\ Z(s+1)_n \end{bmatrix} = \underbrace{\begin{bmatrix} \exp(-t_s/T2)\, R_{\alpha_s,1}^T \cdot \left(F^+(s)_{n-c}, F^-(s)_{n-c}, Z(s)_{n-c}\right) \\ \exp(-t_s/T2)\, R_{\alpha_s,2}^T \cdot \left(F^+(s)_{n+c}, F^-(s)_{n+c}, Z(s)_{n+c}\right) \\ \exp(-t_s/T1)\, R_{\alpha_s,3}^T \cdot \left(F^+(s)_n, F^-(s)_n, Z(s)_n\right) \end{bmatrix}}_{A_{s,n}\, Q(s)_n}$$

$$+\Delta\alpha_s \underbrace{\begin{bmatrix} \exp(-t_s/T2)\, R_{\alpha_s,1}^T \cdot \left(Z(s)_{n-c}, Z(s)_{n-c}, -(F^+(s)_{n-c} + F^-(s)_{n-c})/2\right) \\ \exp(-t_s/T2)\, R_{\alpha_s,2}^T \cdot \left(Z(s)_{n+c}, Z(s)_{n+c}, -(F^+(s)_{n+c} + F^-(s)_{n+c})/2\right) \\ \exp(-t_s/T1)\, R_{\alpha_s,3}^T \cdot \left(Z(s)_n, Z(s)_n, -(F^+(s)_n + F^-(s)_n)/2\right) \end{bmatrix}}_{B_{s,n}\, Q(s)_n}$$

$$\underbrace{+\,\delta_n \begin{bmatrix} 0 \\ 0 \\ M_0\left(1 - \exp(-t_s/T1)\right) \end{bmatrix}}_{d_{s,n}},$$

where the s^{th} gradient imposes c cycles across the voxel.

$$\Rightarrow Q(s+1) = A_s\, Q(s) + \Delta\alpha_s\, B_s\, Q(s) + d_s.$$

The Optimization Problem

Without recovery, the signal progression can be modeled as

$$\hat{Q}(s+1) = \left(A_s + \Delta\alpha_s B_s\right) \hat{Q}(s).$$

The output at time S (approximating $\Delta\alpha_s\, \Delta\alpha_{s'} \approx 0$) is determined recursively:

$$\hat{Q}(S) = \left(\prod_{s=0}^{S-1} A_s\right) Q_0 + \sum_{s=0}^{S-1} \left[\left(\prod_{s'=s+1}^{S-1} A_{s'}\right) B_s \left(\prod_{s'=0}^{s-1} A_{s'}\right) Q_0\, \Delta\alpha_s \right].$$

The signal at time S is the progression of the initial condition summed with the contributions from the progressions of each recovery component:

$$Q(S) = \underbrace{+\,\mathbf{1}^T \begin{bmatrix} \left(\prod_{s=1}^{S-1} A_s\right) B_0 Q_0 & \left(\prod_{s=2}^{S-1} A_s\right) B_1 A_0 Q_0 & \cdots & B_{S-1}\left(\prod_{s=0}^{S-2} A_s\right) Q_0 \\ & \left(\prod_{s=2}^{S-1} A_s\right) B_1 d_1 & \cdots & B_{S-1}\left(\prod_{s=1}^{S-2} A_s\right) d_1 \\ & & \ddots & \vdots \\ & & & B_{S-1} d_{S-1} \end{bmatrix}}_{\boldsymbol{H_S}}.$$

$$\underbrace{\begin{bmatrix} I\Delta\alpha_0 \\ I\Delta\alpha_1 \\ \vdots \\ I\Delta\alpha_{S-1} \end{bmatrix}}_{\boldsymbol{\Delta\alpha}} + \mathbf{1}^T \begin{bmatrix} \left(\prod_{s=0}^{S-1} A_s\right) Q_0 \\ \left(\prod_{s=1}^{S-1} A_s\right) d_1 \\ \vdots \\ A_{S-1}\, d_{S-1} \end{bmatrix} + d_S$$

This can be reformulated as

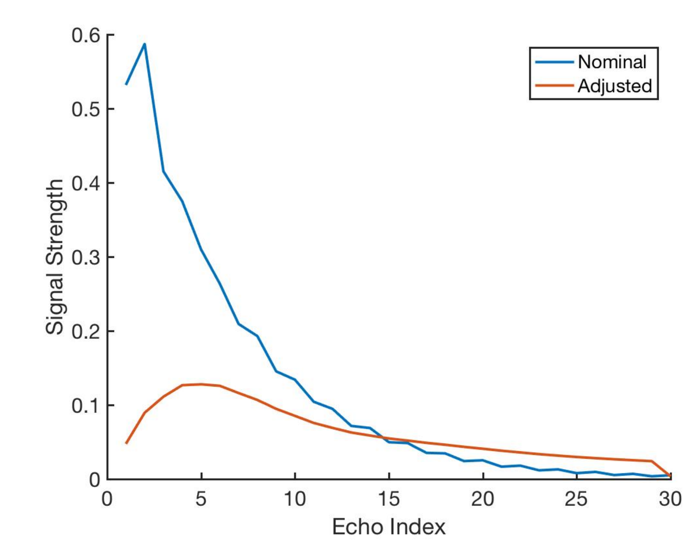
$$\begin{array}{ll} \text{minimize} & \|\boldsymbol{H_S}\, \boldsymbol{\Delta\alpha} - \boldsymbol{b}\|_2 \\ \text{subject to} & \|\boldsymbol{\Delta\alpha}\|_\infty \leq \theta \end{array}$$

where θ is a constraint requiring that $\Delta\alpha$ be small.

If specific states at many times are desired, then the problem can become

$$\begin{array}{ll} \text{minimize} & \|\boldsymbol{H_{S_1}}\, \boldsymbol{\Delta\alpha} - \boldsymbol{b}\|_2^2 + \|\boldsymbol{H_{S_2}}\, \boldsymbol{\Delta\alpha} - \boldsymbol{b}\|_2^2 + \cdots + \|\boldsymbol{H_S}\, \boldsymbol{\Delta\alpha} - \boldsymbol{b}\|_2^2 \\ \text{subject to} & \|\boldsymbol{\Delta\alpha}\|_\infty \leq \theta \end{array}.$$

Example: Smoothing the Signal of FSE



This signal is retained through several more echoes when tip-angles are altered than when compared to a constant tip-angle pulse.

Conclusion

We have developed a formalism that allows for sequence design in a non-greedy fashion when making the small-tip angle approximation.