# Vector Spaces

## Math Lecture 5

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# Binary Operation

**Given a set S, a Binary Operation is a rule for mapping pairs of elements of S to another element of S.**

# External Binary Operation

Given sets S and K, an External Binary Operation is a rule for mapping pairs (k,s) to another element of S.

# Vector Space

A vector space over a field F is a set S with binary operations + and external binary operation x that satisfy the following:

There exists an element 0 such that u + 0 = u
u+v = v+u
(u+v)+w = u+(v+w)
For any u there exists -u such that u + (-u) = 0

There exists a scalar 1 such that 1 x u = u

For any scalar k, k (u + v) = ku + kv
For any scalars k1 and k2, (k1 + k2) u = k1 u + k2 u

# Examples of Vector Spaces

$\mathbb{R}^2$ **Euclidean Plane**

$\mathbb{R}^3$ **Euclidean Space**

$\mathbb{R}^{3000}$

**The set of all functions**

**The set of all continuous functions**

**The set of all polynomials of order 4.**

# Subspace

**A subset of a vector space that is also a vector space is called a subspace.**

**To show that a subset of a Vector Space is a subspace, one must show:**

**It contains the 0 vector**
**It is closed under scalar multiplication**
**It is closed under vector addition**

**Example:**

**Any plane through the origin is a subspace of** $\mathbb{R}^3$

# Basis

**A Basis for a Vector Space V is a set of linearly independent vectors that spans the space.**

**The standard basis is denoted by** $e$

**For** $\mathbb{R}^3$**, the standard basis is** $e = \{e_1, e_2, e_3\}$**.**

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

# Dimension

**The size of any basis for a Vector Space V is the same.**

**The size of the basis is called the Vector Space's Dimension.**

**A Vector Space may not have a finite dimension.**

# Projection Onto Vector Space

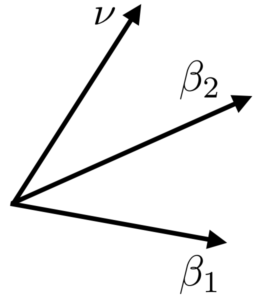
**We've already discussed how to project a vector onto another vector.**

**Now we'll project a vector onto a Vector Space.**

# Projection Onto Vector Space

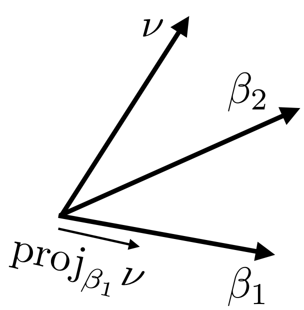
**Let** $\beta = \{\beta_1, \beta_2, \ldots, \beta_N\}$ **be a basis for the Vector Space** *V***.**

**We would like to project** $\nu$ **onto** *V.*



# Projection Onto Vector Space

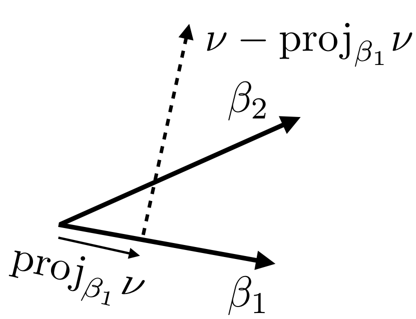
**First, project $\nu$ onto $\beta_1$.**




# Projection Onto Vector Space

**First, project $\nu$ onto $\beta_1$.**

**Subtract $\text{proj}_{\beta_1}\nu$ from $\nu$.**



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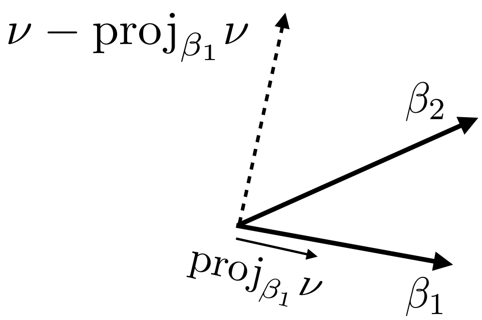
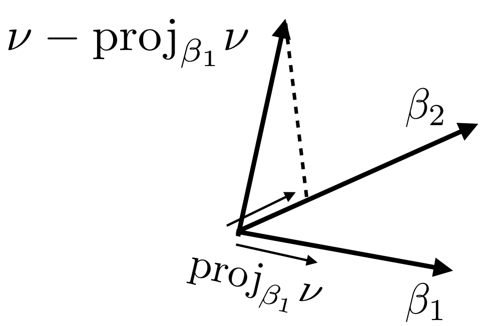



# Projection Onto Vector Space

**First, project $\nu$ onto $\beta_1$.**

**Subtract $\text{proj}_{\beta_1}\nu$ from $\nu$.**

**Project $\nu - \text{proj}_{\beta_1}\nu$ onto $\beta_2$.**



# Projection Onto Vector Space

**The projection of $\nu$ onto the Vector Space $V$ is**

$$\mathrm{proj}_{\beta_1}\nu + \mathrm{proj}_{\beta_2}\left(\nu - \mathrm{proj}_{\beta_1}\nu\right)$$