

Digital Signals

Assignment 2: Due 7/19/2016

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1 Main Problems

Problem 1 *Commutativity of Matrices*

For this problem, consider matrices with real entries.

Part a Either prove that matrices commute with addition or give a counter example.

Part b Either prove that matrices commute with multiplication or give a counter example.

Problem 2 Let $x, y \in \mathbb{R}^3$. Show $x^T y = y^T x$.

Problem 3 *Cross Product as a Linear Transformation* Consider a vector $a \in \mathbb{R}^3$. Find a matrix A such that $Ax = a \times x$ for all $x \in \mathbb{R}^3$.

Problem 4 *Fitting a Polynomial to Data* Suppose you are given a set of N ordered pairs $\{(x_i, y_i) : i = 1, \dots, N\}$. These pairs follow the relationship $y_i = f(x_i)$ where f is a fourth degree polynomial. Show that there exists a matrix A and vector c such that $y = Ac$.

Problem 5 (*Credit: Dr. Boyd*) Consider the rational function (i.e., a ratio of polynomials)

$$f(t) = \frac{c_0 + c_1 t + c_2 t^2}{1 + d_1 t + d_2 t^2}$$

that satisfies the following conditions:

$$f(1) = 2, \quad f(2) = 5, \quad f(3) = 9, \quad f(4) = -2, \quad f(5) = -4.$$

Find the numbers c_0, c_1, c_2, d_1 , and d_2 for which these conditions hold.

Part a Let $x = (c_0, c_1, c_2, d_1, d_2)$. Explain how to formulate this problem as finding a vector x that satisfies a system of linear equations $Ax = b$. Be sure to specify what A and b are.

Part b Solve the system of linear equations with Python. What do you get for vector x ?

Problem 6 (*Credit: Dr. Boyd*) In this puzzle we consider several linear functions of a grayscale image with $N \times N$ pixels. To keep the matrices small enough to work out by hand, we will consider the case when $N = 3$ (which would hardly qualify as an image). We represent a 3×3 image as a vector in \mathbb{R}^9 using the ordering of pixels shown below. (This is called *column-major* ordering.)

X_1	X_4	X_7
X_2	X_5	X_8
X_3	X_6	X_9

Each of the operations or transformations below defines a function $y = f(x)$, where x is the original 9-vector, and y is the 9-vector that corresponds to the transformed image. For each of the following operations, give the 9×9 matrix A for which $y = Ax$.

Part a Turn the original image upside down.

Part b Rotate the original image clockwise by $\frac{\pi}{2}$ radians.

Part c Translate the image up by 1 pixel and right by 1 pixel. In the translated image, assign the value $y_i = 0$ to the pixels in the first column and the last row.

Part d Set each pixel y_i to be the average of the neighbors of pixel i in the original image. By neighbors, we mean the pixels immediately above and below, and immediately to the left and right. The center pixel has 4 neighbors, corner pixels have two neighbors, and the remaining pixels have 3 neighbors.

Problem 7 *Signal Detection in Noise* A communication system may be in standby mode until needed; this means that its main systems are shut down, and only auxiliary power is used to determine if it needs to turn on the whole system or not. This is common with satellites and cellular phone communications. In order to determine if it needs to wake up, it may be looking for a specific signal embedded in all of the noise. A simple way to do this is with the use of cross-correlation.

Suppose that $y \in \mathbb{R}^M$ represents a short signal that notifies the device to exit standby mode. Suppose that $x \in \mathbb{R}^N$ represents a stream of data where the signal y may be embedded. Note that usually $N \gg M$. The cross correlation of y and x is

$$(y \star x)_n = \sum_{m=-\infty}^{\infty} y_m x_{m+n}.$$

It is assumed that all values for indices of y and x that aren't valid are 0. Thus, $y \star x \in \mathbb{R}^N$. Effectively, the signal x is "slid along" the signal y . At each point, a dot product is performed between y and the overlapping region of x .

Hint: To understand the above equation, try figuring out what happens when $n = 0$. Then figure out when $n = 1$, then $n = 2$. Do you see the way it works?

For this problem, we will play with the following toy problem. Define $y = (0.8, 0.5, 0.1, 0.1, 0.8)$. Find a matrix A such that $Ax = y \star x$ for all x .

Note, there exists a very related technique called *Normalized Cross Correlation*; this technique is used throughout engineering as a very reliable way to detect signals in noise. In fact, I used it as part of a urinalysis device that was recently published and in the news!¹

Problem 8 A linear function f is one that satisfies the following two conditions:

- $f(x + y) = f(x) + f(y)$
- $k f(x) = f(kx)$

for all vectors x and y and scalars k . Consider a linear function $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$. There exists a matrix A such that $f(x) = Ax$ for all x .

Part a What size is A ?

Part b Find A (in terms of f applied to specific vectors). *Hint: consider $f((1, 0, 0, \dots, 0))$.*

Problem 9 There is a way to embed a message in an image. Very soon, some students will find you; they will be relying on you to decode a message for them so that they can progress in their game. They will present you with an image and a matrix and ask you for the message. The fate of the world rests on your shoulders.

Let X denote the original image. The variable x is a vector formed by column extending the image; it has N elements. We will alter it as follows: $y = x + As$, where A is a matrix and s is the message. The matrix A is of size $N \times S$, where S is the number of symbols in the message. We assume that the message is binary code, so s is a vector only of values 0 or 1. The entries of A are small so that $y \approx x$.

To recover s , we use the pseudo-inverse of the matrix A , denoted as A^\dagger . Let

$$\tilde{s} = A^\dagger y = A^\dagger (x + As) = A^\dagger x + A^\dagger As = A^\dagger x + s.$$

The value $A^\dagger x$ is noise, and we hope that it's small compared to our message s . We form our estimate of the code as $\hat{s} = \text{round}(\tilde{s})$.

Write a code that extracts the image from the matrix. Your function prototype should be as follows:

```
def extractMessage( img, A )
    # relevant code goes here
    return message
```

There is a sample image to do this with at

www.stanford.edu/~ndwork/si2016/session2/hmwk2/embedded.png

The matrix A that encoded the message is in the following file:

www.stanford.edu/~ndwork/si2016/session2/hmwk2/encodeMatrix.npy

¹<http://www.cnet.com/news/want-to-test-your-pee-for-diseases-at-home-urinalysis/>

I've written some skeleton code for you to read in the matrix and the image from the file. You need to figure out how to decode the message. You can get the skeleton code here:

`www.stanford.edu/~ndwork/si2016/session2/hmwk2/decodeMessage.py`

The message has been represented using Huffman encoding. (If I've forgotten to explain this to you, please remind me so that I can teach this in class.) You can find the Huffman encoding tree here:

`www.stanford.edu/~ndwork/si2016/session2/hmwk2/huffCode.png`

What message do you get out?

Problem 10 *Consumption and Demand* The Leontief Input Output Model describes the consumption and production of a group of industries. The *Consumption Matrix* C is defined so that C_{ij} is the output from industry i (in units of money) that is needed to produce output (in units of money) from industry j . Suppose there are n industries, and each has produced quantity p_i (in units of money).

Part a Find an expression for the amount of goods consumed by the n industries in the next iteration of production.

Part b Find an expression for the net goods produced by the n industries in the next iteration of production.

Problem 11 Create a matrix of zeros 500×500 in size called "myImage". Then execute the following command:

```
myImage[1:100,1:200] = 255
```

Display this image on the screen. What does this show you about image coordinates in Python? (Provide as many conclusions as you can.)

2 Challenge Problems

CP 1 Let a and b be the following vectors:

$$a = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \quad b = \begin{bmatrix} 3 \\ 8 \end{bmatrix}.$$

The vector b can be written as the sum of two vectors, one that's parallel to a and one that's perpendicular to a . Find these vectors. (Note: The vector that is parallel to b is called "the projection of a onto b ".)

CP 2 The reverse triangle inequality states that for any two vectors a and b ,

$$|||a||_2 - ||b||_2| \leq ||a - b||_2.$$

Prove this.

CP 3 A field of math is defined by a set of axioms. Axioms are statements that we accept as true without proof. The axioms of the real numbers are as follows:

- There exists a number 0 such that for any number a , $a + 0 = a$.
- There exists a number 1 such that for any number a , $a \cdot 1 = a$.
- For any number a , there exists a number $(-a)$ such that $a + (-a) = 0$.
- For any number $a \neq 0$, there exists a number a^{-1} such that $a \cdot a^{-1} = 1$.
- For any two numbers a, b , $a + b = b + a$.
- For any two numbers a, b , $ab = ba$.
- For any three numbers a, b, c , $(a + b) + c = a + (b + c)$.
- For any three numbers a, b, c , $(ab)c = a(bc)$.
- For any numbers a, b, c , $a(b + c) = ab + ac$.

A theorem is a statement that we can prove given the axioms. Using only the axioms listed above, prove the following theorem:

- For any number a , $a \cdot 0 = 0$.

CP 4 Write a function that accepts a natural number and outputs a triangle of numbers so that the elements in each column indicate the reverse column id. The prototype of the function should be as follows:

```
def makeNumberTriangle( N )
```

A sample output where the input number $N = 4$ is shown below

```

          1
         2 1
        3 2 1
       4 3 2 1
```

CP 5 *Where Am I?*

On older aircraft, or if a GPS system doesn't work, pilots rely on Very High Frequency Omnidirectional Ranging (VOR) systems to identify their location. VOR systems near aircraft send out electromagnetic signals; the receiving system on board the aircraft is able to determine the direction that the signal is coming from (but not the distance to the source of the signal). By receiving signals from multiple VOR systems, the pilot can triangulate to determine the location of his/her aircraft. And that's exactly what you'll do in this problem.

Your aircraft is receiving signals from three different VOR stations. The following table describes the position of the VOR stations and the angle that the signal is coming from, where the direction is the angle from the East (i.e., 0° is directly east and 90° is directly north).

VOR	East (meters)	North (meters)	Direction (degrees)
A	-170	760	119.3
B	1290	150	14.9
C	490	-1100	-80.4

Part a Find a matrix A and vector b so that (if there were no noise) the position vector of the aircraft x would satisfy $Ax = b$.

Part b Use some computer system to find an estimate of the position by finding the vector that minimizes $\|Ax - b\|_2$. A good tool to do this might be wolframalpha.com. Try searching for “wolfram alpha least squares” with Google to figure out how.