Signal Processing and Linear Systems 1

Matrices and Vectors Review

<u>Matrix Vector Multiplication</u>

- *A* is an *m* x *n* dimensional matrix
- *x* is a *n* dimensional vector
- *y* is a *m* dimensional vector

$$y = Ax$$

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1} & A_{m2} & \dots & A_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Matrix Vector Multiplication

$$y_i = \sum_{j=1}^{n} A_{ij} x_j \text{ for } i = 1, \ldots, m$$

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \ldots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1} & A_{m2} & \ldots & A_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Matrix Vector Multiplication

Interpretation 1: weighted sum of columns of A

$$y = \begin{bmatrix} | & & | \\ a_1 & \ldots & a_n \\ | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$y = \sum_{j=1}^{n} x_j a_j \text{ where } a_j \text{ is the } j\text{th column of } A$$

Matrix Vector Multiplication

Interpretation 2: inner product with rows

$$A = \begin{bmatrix} \tilde{a}_1^T \\ \tilde{a}_2^T \\ \vdots \\ \tilde{a}_n^T \end{bmatrix} \qquad y = \begin{bmatrix} \tilde{a}_1^T x \\ \tilde{a}_2^T x \\ \vdots \\ \tilde{a}_m^T x \end{bmatrix}$$

Matrix Vector Multiplication (example)

$$\begin{bmatrix} 1 & 2 \\ 3 & -1 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Interpretation 1: weighted sum of columns of A :

$$= 2 \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 4 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 2+2 \\ 6-1 \\ 4+4 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 8 \end{bmatrix}$$

Interpretation 2: inner product with rows:

$$= \begin{bmatrix} (1,2) \cdot (2,1) \\ (3,-1) \cdot (2,1) \\ (2,4) \cdot (2,1) \end{bmatrix} = \begin{bmatrix} 2+2 \\ 6-1 \\ 4+4 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 8 \end{bmatrix}$$

Eigenvalues and Eigenvectors

Sometimes when we multiply a square matrix by a vector the result is just the original vector scaled by a constant value.

$$Av = \lambda v$$

For a given matrix *A*, any vector *v* that satisfies this relationship is an eigenvector of *A*, associated with eigenvalue λ

Eigenvalues and Eigenvectors (example)

$$Av = \lambda v$$

$$\begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ 14 \end{bmatrix} = 7 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \text{ is an eigenvector of } \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix} \text{ associated with eigenvalue } 7$$

Representing a Linear Transformation

Any linear transformation can be represented as matrix multiplication. A transformation $f : \boldsymbol{R}^n \to \boldsymbol{R}^m$ is linear if it satisfies:

$$1)\ \text{homogeneity}\ \ f(\alpha x) = \alpha f(x)$$

$$2)\ \text{additivity}\ \ f(x+y) = f(x) + f(y)$$

Representing a Linear Transformation

$$f(x) = Ax$$

where the $j$the column of A is $f(e_j)$. e_j is the jth standard basis vector, that is the vector for which the jth element is 1 and all other elements are 0.

$$f(x) = \begin{bmatrix} | & & | \\ f(e_1) & \dots & f(e_n) \\ | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

Representing a Linear Transformation (example)

$$f(x_1, x_2) = (x_1 + 2x_2, 3x_1 - x_2, x_1 + x_2)$$

$$f(x) = \begin{bmatrix} | & & | \\ f(e_1) & \dots & f(e_n) \\ | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$f(e_1) = f(1,0) = (1,3,1) \qquad f(e_2) = f(0,1) = (2,-1,1)$$

$$f(x_1, x_2) = \begin{bmatrix} 1 & 2 \\ 3 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Image Credits

Stanford Engineering Everywhere, EE263, Stephen Boyd