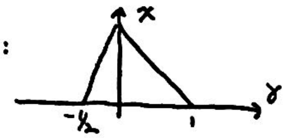
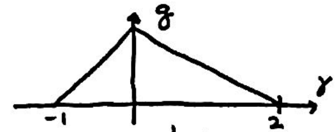
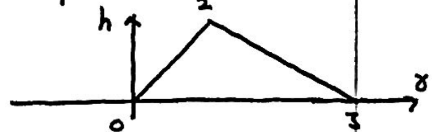
Lecture Notes for 7/7.

§ Review of Scaling and Shifting.

Ex:  Plot $x\left(\frac{\gamma-1}{2}\right)$.

Method 1: Let $g(\gamma) = x\left(\frac{\gamma}{2}\right)$. 

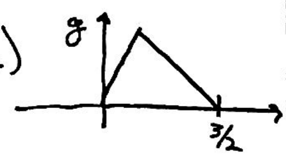
~~Method 2:~~ Let $h(\gamma) = g(\gamma-1) = x\left(\frac{\gamma-1}{2}\right)$. 

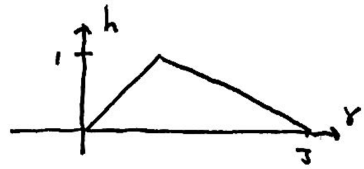
Method 2: $x\left(\frac{\gamma-1}{2}\right) = x\left(\frac{\gamma}{2} - \frac{1}{2}\right)$.

Let $g(\gamma) = x\left(\frac{\gamma}{2}\right)$. Same as before.

Let $h(\gamma) = g(\gamma - \frac{1}{2}) = x\left(\frac{\gamma}{2} - \frac{1}{2}\right)$. 

What went wrong?

Method 2 (correctly): Let $g(\gamma) = x(\gamma - \frac{1}{2})$ 

Let $h(\gamma) = g(\gamma/2)$. 

§ Cascading Systems

$$y \to \boxed{S_1} \to \boxed{S_2} \to \cdots \to \boxed{S_N} \to x$$

In general, $S_N\{\cdots S_2\{S_1\{y\}\}\} = y$.

If S_i is LSI for all i,

$$x = (((y * h_1) * h_2) * \cdots * h_N),$$

where $h_i = S_i\{\delta\}$ for all i.

Convolution is hard. We have found away to represent signals as linear combinations of transcendental functions. Hopefully this will help us.

§ Fourier Series

If either $f : [0,T] \to \mathbb{C}$ or f is periodic with period T then we can represent f as a Fourier Series:

$$f(x) = \sum_{n=-\infty}^{\infty} F_n \exp\left(i 2\pi \frac{nx}{T}\right)$$

where the Fourier Coefficients are

$$F_n = \frac{1}{T} \int_0^T f(x) \exp\left(-i 2\pi \frac{nx}{T}\right) dx .$$

If $f : \mathbb{R} \to \mathbb{C}$ then we can represent f as a Fourier Transform

$$f(x) = \int_{-\infty}^{\infty} F(k) \exp(i 2\pi k x)\, dk$$

where $$F(k) = \int_{-\infty}^{\infty} f(x) \exp(-i 2\pi k x)\, dx.$$

Equivalent Notation: $F = \mathcal{F}\{f\}$ $\quad f \to \boxed{\mathcal{F}} \to F$

$f = \mathcal{F}^{-1}\{F\}$. $\quad F \to \boxed{\mathcal{F}^{-1}} \to f$

Ex 1: $f(x) = \delta(x)$.

$$F(s) = \int_{-\infty}^{\infty} \delta(x)\, e^{-i 2\pi s x}\, dx = e^0 = 1.$$

$$\therefore \mathcal{F}\{\delta\} = 1.$$

Ex: $f(x) = \Pi(x)$.

$$F(s) = \int_{-\infty}^{\infty} \Pi(x)\, e^{-i2\pi s x}\, dx = \int_{-1/2}^{1/2} e^{-i2\pi s x}\, dx.$$

Case 1: $s = 0 \Rightarrow F(s) = \int_{-1/2}^{1/2} dx = 1$.

Case 2: $s \neq 0$

$$F(s) = \frac{-1}{i2\pi s} e^{-i2\pi s x}\Big|_{-1/2}^{1/2} = \frac{-1}{i2\pi s}\left(e^{-i\pi s} - e^{i\pi s}\right)$$

$$= \frac{\sin(\pi s)}{\pi s}.$$

$$\therefore F(s) = \begin{cases} 1 & \text{if } s = 0 \\ \dfrac{\sin(\pi s)}{\pi s} & \text{otherwise} \end{cases} \qquad F(s) = \operatorname{sinc}(s).$$

Ex: $f(x) = e^{-ax} u(x)$.

$$F(s) = \int_{-\infty}^{\infty} f(x)\, e^{-i2\pi s x}\, dx = \int_{-\infty}^{\infty} e^{-ax} u(x)\, e^{-i2\pi s x}\, dx$$

$$= \int_{0}^{\infty} e^{-ax} e^{-i2\pi s x}\, dx = \int_0^{\infty} e^{-(a+i2\pi s)x}\, dx$$

$$= \frac{-1}{a+i2\pi s} e^{-(a+i2\pi s)x}\Big|_0^{\infty} = \frac{1}{a+i2\pi s}$$

Ex: $f(x) = u(x)$. Note: $f(x) = \lim_{a\to 0} e^{-ax} u(x)$.

$$\Rightarrow F(s) = \frac{1}{i2\pi s}.$$