Assignment 6 Solutions

1/ Claim: $\mathrm{DFT}\{a \circledast b\} = \mathrm{diag}(\mathrm{DFT}\{a\})\,\mathrm{DFT}\{b\}$.

Proof:

$$\mathrm{DFT}\{a \circledast b\} = \mathrm{DFT}\left\{ \sum_{m=0}^{N-1} a[m]\, b[n-m] \right\}$$

where indexing is done mod N.

$$\Rightarrow \mathrm{DFT}\{a \circledast b\}[k] = \sum_{n=0}^{N-1} \left(\sum_{m=0}^{N-1} a[m]\, b[n-m] \right) e^{-i2\pi \frac{kn}{N}}$$

$$= \sum_{m=0}^{N-1} a[m] \sum_{n=0}^{N-1} b[n-m]\, e^{-i2\pi \frac{kn}{N}}$$

$$= \sum_{m=0}^{N-1} a[m]\, B[k]\, e^{-i2\pi \frac{km}{N}}$$

$$= B[k] \sum_{m=0}^{N-1} a[m]\, e^{-i2\pi \frac{km}{N}} = A[k]\, B[k]. \quad \blacksquare$$

3/ a) Show $\tilde{\mathcal{F}}^{-1}\{f * g\} = \tilde{\mathcal{F}}^{-1}\{f\}\, \tilde{\mathcal{F}}^{-1}\{g\}$.

$$\tilde{\mathcal{F}}^{-1}\{f * g\}(x) = \int_{-\infty}^{\infty} (f*g)(k)\, e^{i2\pi kx}\, dk$$

$$= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(\gamma)\, g(k-\gamma)\, d\gamma \right) e^{i2\pi kx}\, dk$$

$$= \int_{-\infty}^{\infty} f(\gamma) \int_{-\infty}^{\infty} g(k-\gamma)\, e^{i2\pi kx}\, dk\, d\gamma$$

$$= \int_{-\infty}^{\infty} f(\gamma)\, \hat{g}(k)\, e^{i2\pi k\gamma}\, d\gamma = \hat{f}(k)\, \hat{g}(k),$$

where $\hat{f} = \tilde{\mathcal{F}}^{-1}\{f\}$ and $\hat{g} = \tilde{\mathcal{F}}^{-1}\{g\}$.

b) Show $\tilde{\mathcal{F}}\{fg\} = \tilde{\mathcal{F}}\{f\} * \tilde{\mathcal{F}}\{g\}$.

The above is true if and only if

$$\tilde{\mathcal{F}}^{-1}\{\tilde{\mathcal{F}}\{fg\}\} = \tilde{\mathcal{F}}^{-1}\{\tilde{\mathcal{F}}\{f\} * \tilde{\mathcal{F}}\{g\}\}$$

$$\Leftrightarrow fg = \tilde{\mathcal{F}}^{-1}\{\tilde{\mathcal{F}}\{f\}\}\, \tilde{\mathcal{F}}^{-1}\{\tilde{\mathcal{F}}\{f\}\} \text{ by part (a)}$$

$$\Leftrightarrow fg = fg, \text{ which is true.}$$

4) The first circuit is a high pass circuit.

$$H(k) = \frac{Z_{c_1} Z_{c_2}}{R_1 R_2 + Z_{c_1}(R_1 + R_2) + Z_{c_1} Z_{c_2}} \qquad Z_{c_1} = \frac{1}{j2\pi k C_1} \qquad Z_{c_2} = \frac{1}{j2\pi k C_2}.$$

The second circuit is a low pass circuit.

$$H(k) = \frac{R_1 R_2}{Z_{c_1} Z_{c_2} + R_1 (Z_{c_1} + Z_{c_2}) + R_1 R_2}.$$

5) $$\sum_{m=0}^{M-1} a_m y^{(m)}(t) = \sum_{n=0}^{N-1} b_n x^{(n)}(t).$$

Claim: this system is linear.

Proof:

w.t.s. if (x_1, y_1) and (x_2, y_2) are solutions then $(x_1 + x_2, y_1 + y_2)$ is a solution.

$$\sum_{m=0}^{M-1} a_m (y_1 + y_2)^{(m)}(t) = \sum_{m=0}^{M-1} a_m (y_1^{(m)} + y_2^{(m)})(t)$$
$$= \sum_{m=0}^{M-1} a_m y_1^{(m)}(t) + \sum_{m=0}^{M-1} a_m y_2^{(m)}(t) = \sum_{n=0}^{N-1} b_n x_1^{(n)}(t) + \sum_{n=0}^{N-1} b_n x_2^{(n)}(t)$$
$$= \sum_{n=0}^{N-1} b_n (x_1 + x_2)^{(n)}(t).$$

w.t.s. if (x, y) is a solution then $(\alpha x, \alpha y)$ is a solution.

$$\sum_{m=0}^{M-1} a_m (\alpha y)^{(m)}(t) = \alpha \sum_{m=0}^{M-1} a_m y^{(m)}(t) = \alpha \sum_{n=0}^{N-1} b_n x^{(n)}(t)$$
$$= \sum_{n=0}^{N-1} b_n (\alpha x)^{(n)}(t).$$

∴ the system is linear.

Claim: the system is shift invariant.

Proof:

w.t.s. if $(x(t), y(t))$ is a solution then $(x(t-\Delta), y(t-\Delta))$ is a solution.

$$\sum_{m=0}^{M-1} a_m y^{(m)}(t-\Delta) = \sum_{m=0}^{M-1} a_m y^{(m)}(u) = \sum_{n=0}^{N-1} b_n x^{(n)}(u)$$
$$= \sum_{n=0}^{N-1} b_n x^{(n)}(t-\Delta), \quad \text{where } u = t - \Delta.$$

∴ the system is shift invariant.

6) From the circuit diagram, we see

$$\int\left[\int(x-by)-ay\right]=y.$$

Taking derivatives:

$$\int(x-by)-ay=y'$$

$$x-by-ay'=y'' \Rightarrow \underline{x=y''+ay'+by}.$$