Assignment 5 - Solutions

1/ a) Find $\int_{-\infty}^{\infty} |\text{sinc}(x)|^2 \, dx$.

By Parseval's Theorem,

$$\int_{-\infty}^{\infty} |\text{sinc}(x)|^2 \, dx = \int_{-\infty}^{\infty} |\tilde{\mathfrak{F}}\{\text{sinc}\}(k)|^2 \, dk$$

$$= \int_{-\infty}^{\infty} \Pi^2(k) \, dk = 1.$$

b) Find $\int_{-\infty}^{\infty} \text{sinc}(x) \, dx$.

$$\tilde{\mathfrak{F}}\{\text{sinc}\}(k) = \int_{-\infty}^{\infty} \text{sinc}(x) \, e^{-i2\pi x k} \, dx.$$

$$\Rightarrow \int_{-\infty}^{\infty} \text{sinc}(x) \, dx = \tilde{\mathfrak{F}}\{\text{sinc}\}(0) = \Pi(0) = 1.$$

2/ One example of such a function is any constant function $f(t) = c$, where $c \in \mathbb{R}$.

A more interesting function is

$$f(t) = \begin{cases} \cos(2\pi/t) & t \neq 0 \\ 0 & t = 0 \end{cases}.$$

3/ a) The output of a Linear Shift Invariant system equals

$$y = x * h$$

where h is the impulse response.

When $x(t) = \exp(i2\pi f t)$,

$$y(t) = \int_{-\infty}^{\infty} h(\gamma) \exp\left(i2\pi f (t-\gamma)\right) d\gamma$$

$$= \exp(i2\pi f t) \int_{-\infty}^{\infty} h(\gamma) \, e^{-i2\pi f \gamma} \, d\gamma$$

$$= H(f) \exp(i2\pi f t).$$

This shows that $x(t) = \exp(i2\pi f t)$ is an eigenfunction of any LSI system, and $H(f)$ is its corresponding eigenvalue (where $H = \tilde{\mathfrak{F}}\{f\}$).

b) Consider $f(t) = (g * h)(t)$.

$$g(t) = \int_{-\infty}^{\infty} G(k)\, e^{i2\pi kt}\, dt.$$

$$\Rightarrow f(t) = \left(\int_{-\infty}^{\infty} G(k)\, e^{i2\pi kt}\, dt\right) * h(t)$$

$$= \int_{-\infty}^{\infty} G(k) \left[e^{i2\pi kt} * h(t)\right] dt \quad \text{since } * \text{ is linear}$$

$$= \int_{-\infty}^{\infty} G(k)\, H(k)\, e^{i2\pi kt}\, dt \quad \text{from part (a).}$$

$$\therefore F(k) = \mathcal{F}\{f\}(k) = G(k)H(k). \qquad \blacksquare$$

5/ a) $\mathcal{F}\{x * y\} = X\,Y. \Rightarrow B = 1/2.$

b) $\mathcal{F}\{xy\} = X * Y \Rightarrow B = 3/2.$

c) $f = x * u. \Rightarrow F = X \cdot \mathcal{F}\{u\}.$

$$\mathcal{F}\{u\}(k) = 1/2 \left[\delta(k) - \frac{i}{\pi k}\right].$$

Since the support of $\mathcal{F}\{u\}$ is $\mathbb{R}$, the bandwidth of F equals that of X, which is $1/2$.

d) $g(\gamma) = x(\gamma)\cos(2\pi\gamma).$

$$\Rightarrow G(k) = X(k) * 1/2\left(\delta(k-1) + \delta(k+1)\right)$$

$$= 1/2\left[X(k-1) + X(k+1)\right].$$

$$\therefore B = 3/2.$$

e) $f(\nu) = x(\nu)\, e^{i2\pi\nu} \Rightarrow F(k) = X(k) * \delta(k+1).$

$$\therefore B = 3/2.$$

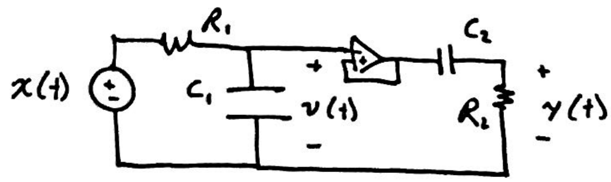
f) $g = (x * y)'. \Rightarrow G(k) = (i2\pi k)\, \mathcal{F}\{x * y\}(k)$

$$= (i2\pi k)\, X(k)\, Y(k).$$

$$\therefore B = 1/2.$$

6/ $$D = \left.\underbrace{\begin{bmatrix} -1 & 1 & & & \\ & -1 & 1 & & \\ & & \ddots & \ddots & \\ & & & -1 & 1 \end{bmatrix}}_{N \text{ columns}}\right\} N-1 \text{ rows}$$

7/ Review the previous homework for the transfer functions of the first 3 circuits.



$$V(k) = \frac{Z_{c1}\, X(k)}{Z_{c1} + R_1}, \quad \text{where } Z_{c1}(k) = \frac{1}{j2\pi k C_1}.$$

Since the voltage at both input terminals of the operational amplifier are equal,

$$Y(k) = \frac{R_2}{Z_{c2} + R_2} V(k) \quad \text{where } Z_{c2}(k) = \frac{1}{j2\pi k C_2}.$$

$$\therefore\ H(k) = \frac{Y(k)}{X(k)} = \left(\frac{Z_{c1}(k)}{Z_{c1}(k) + R_1}\right)\left(\frac{R_2}{Z_{c2}(k) + R_2}\right).$$

8/ $\omega(s) = -2\pi s.$

$$\mathcal{F}^{-1}\{F\}(x) = \int_{-\infty}^{\infty} F(s)\, e^{i2\pi s x}\, ds$$
$$= \int_{-\infty}^{\infty} F\left(\frac{\omega}{-2\pi}\right) e^{-i\omega x}\, ds$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{F}(\omega)\, e^{-i\omega x}\, d\omega.$$

9) $\int_{-\infty}^{\infty} f(x)\,\bar{g}(x)\,dx = \int_{-\infty}^{\infty} f(x)\,\mathfrak{F}^{-1}\{G(k)\}\,dx$

$$= \int_{-\infty}^{\infty} f(x) \int_{-\infty}^{\infty} G(k)\, e^{i2\pi kx}\, dk\, dx$$

$$= \int_{-\infty}^{\infty} \overline{G}(k) \int_{-\infty}^{\infty} f(x)\, e^{-i2\pi kx}\, dx\, dk$$

$$= \int_{-\infty}^{\infty} \overline{G}(k)\, F(k)\, dk. \quad \blacksquare$$